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The problem of describing the generators of a differential field of invariant differential rational functions with respect to the action of the group of real representations of symplectic transformations in a quaternion space

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Kvaternion fazoning simplektik almashtirishlarini haqiqiy tasvirlari gruppasi ta'siriga nisbatan invariant differensial ratsional funksiyalar differensial maydonining tashkil etuvchilarini tavsiflash masalasi

Ushbu maqola n o'lchovli kvaternion fazoning simplektik almashtirishlarining haqiqiy tasvirlari gruppasi ta'siriga nisbatan invariant differensial ratsional funksiyalarning differensial maydonini tashkil etuvchilarini tavsiflash va ular orasidagi munosabatlarni aniqlashga bag'ishlangan.

Kalit so'zlar: haqiqiy tasvirlar gruppasi; invariant ko'phad; invariant ratsional funksiya; differensial halqa; differensial maydon.

Задача описания образующих дифференциального поля инвариантных дифференциальных рациональных функций относительно действия группы вещественных представлений симплектических преобразований в кватернионном пространстве

Настоящая статья посвящена решению задачи об описание образующие дифференциального поля инвариантных дифференциальных рациональных функций относительно действия группы вещественных представлений симплектических преобразований $n-$мерного кватернионного пространства и определеную соотношения между ними.

Ключевые слова: группа вещественных представлений; инвариантный многочлен; инвариантная рациональная функция; дифференциальное кольцо; дифференциальное поле.

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Introduction

It is known, the classical theory of invariants deals with the classification of elements of a finite-dimensional vector space V over the field k with respect to the action of the algebraic linear group $G \subset GL(n,k)$. In this case, invariants are not arbitrary functions on V these are constants on the orbits of the group G, but only polynomial

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ones (that is written in coordinates as polynomials) or, more generally, rational ones. The polynomial invariants of the group G form a subalgebra in the algebra $k[V]$ of all polynomial functions on V. This subalgebra is called *the algebra of invariants* and is denoted by $k[V]^G$. In the course of the theory invariants, *the problems of describing the generators* of the algebra $k[V]^G$ (which will not necessarily be a complete system of invariants) and *finding the defining relations between them* are considered. The solutions to these problems are called the *first* and *second main theorems* of the theory of invariants respectively(see [1]). In particular, the problem related to the finite generation of the algebra $k[V]^G$ is known as Hilbert's 14th problem. This problem was solved positively by the Hilbert-Nagata-Mumford theorem for many algebraic linear groups, including reductive groups (see [2], [3], [4], [5]). However, in general, i.e., for any algebraic linear group $G \in GL(V)$, the problem is not solved positively (for example, see [2], [3]).

In the study of Hilbert's 14th problem, H.Weyl's works [6] is commendable. In this work, he showed the first and second main theorems for many classical groups, in particular, *linear, special linear, orthogonal and symplectic groups*, methods for proving them, and the branches of development of the theory invariants in later periods.

The differential analogue of the above problems were studied by Dj. Khadzhiev, K. K. Muminov, I. V. Chilin, R. G. Aripov, and obtained the positively solutions of this problem with respect to the action of *orthogonal,special orthogonal, pseudo-orthogonal, and symplectic groups*, (see [7, 8, 9, 10]). At present, the results obtained are applied to differential geometry, non-Euclidean geometry and other important fields of science (see [11, 12, 13, 14, 15]).

In all the works listed above, the posed problem was studied for finite-dimensional real and complex spaces. It is known that in the classical theory of invariants, in addition to real and complex spaces, linear spaces over the skew-field of quaternion numbers are also considered and invariants with respect to the action of subgroups invertible linear transformation in such spaces are studied. This is represented a special case of the theory of non-commutative invariants (see, for example, [16, 17]).

In this paper it will be considered the problems of describing the generators of a differential field of invariant rational functions with respect to the action of the group real representations of symplectic transformations in an n-dimensional quaternion vector space, and that defining relations between them.

Symplectic group

Let H^n be an n-dimensional linear space over the skew-field H (multiplication of numbers is defined on the left), where H is a skew-field of quaternion numbers. By $GL(H^n)$ denote the group of all invertible linear transformations of the space H^n. Let the function $\langle x, y\rangle$ be a mapping of the Cartesian product $H^n \times H^n$ onto H and satisfies the following conditions:

$$\begin{cases} \langle \alpha x + \beta y, z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle, \ \alpha, \beta \in H, \ x, y, z \in H^n; \\ \langle x, y\rangle = \overline{\langle y, x\rangle}; \\ \langle x, x\rangle > 0, \texttt{for every}\, x \in H^n, \ x \neq 0, \end{cases} \tag{1}$$

where $\bar{q}$ means the conjugate of a quaternion $q = a + bi + cj + dk$, i.e., $\bar{q} = a - bi - cj - dk$; $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$; $a, b, c, d \in R$.

We define the function $\langle x, y\rangle$ as a bilinear form as follows:

$$\langle x, y\rangle = x_1\bar{y}_1 + x_2\bar{y}_2 + ... + x_n\bar{y}_n. \tag{2}$$

It is known, the symplectic group $Sp(n)$ with respect to the function $\langle x, y\rangle$ is defined as a subgroup of $GL(H^n)$ as follows, [19]:

$$Sp(n) = \{\sigma \in GL(H^n) : \langle \sigma x, \sigma y\rangle = \langle x, y\rangle\}. \tag{3}$$

It is plain that for $\forall x \in V$ and $\forall \sigma \in GL(H^n)$ the relation $\sigma x \leftrightarrow xg$ is true, where $g \in GL(n, H)$. In this case, the symplectic group $Sp(n)$ is defined as follows

$$Sp(n) = \{g \in GL(n, H) : g^* g = g g^* = E\}, \tag{4}$$

where the matrix g^* is Hermitian conjugate of the matrix g, i.e., $g^* = \bar{g}^T$, E is identity element of the group $GL(n, H)$.

It is known that the space H^n can be considered as to a $4n$ dimensional real space using the following operation:

$$\begin{gathered} x=(x_1,x_2,...,x_n)=(x_{11}+x_{12}i+x_{13}j+x_{14}k,...,x_{n1}+x_{n2}i+x_{n3}j+x_{n4}k)= \\ =(x_{11}+x_{12}i+x_{13}j+x_{14}k)\,e_1+...+(x_{n1}+x_{n2}i+x_{n3}j+x_{n4}k)\,e_n= \\ =x_{11}e_1+x_{12}\,(ie_1)+x_{13}\,(je_1)+x_{14}\,(ke_1)+...+x_{n1}e_n+x_{n2}\,(ie_n)+x_{n3}\,(je_n)+x_{n4}\,(ke_n)\approx \\ \approx(x_{11},x_{12},x_{13},x_{14},...,x_{n1},x_{n2},x_{n3},x_{n4})=\vec{x}, \end{gathered} \tag{5}$$

where $x_{lm}\in R$, $l=\overline{1,n}$, $m=\overline{1,4}$.
We conditionally call the *realification* of this operation and denoted by "$\approx$". We are denoted by V the space of the realification H^n.

It is obvious that as a result of applying the operation "$\approx$ the sum of arbitrary vectors $x,y\in H^n$ turns into the sum of vectors $\vec{x},\vec{y}\in V$, where $\vec{x},\vec{y}$ are real vectors corresponding to the vectors x,y. However, this property does not hold for the operation of the multiplication (on the left) of an arbitrary vector $x\in H^n$ by a number $\lambda\in H$. Therefore, when realification the space, the concepts associated with the operation of scalar multiplication are defined with the help of certain conditions through their equivalent concepts. For example, *linearly dependent, orthogonally and other*. Accordingly, we introduce the following definition.

Definition 1. Vectors $\vec{x}_1,\vec{x}_2,...,\vec{x}_n\in V$ are called a *strongly linearly independent* if the vectors $x_1,x_2,...,x_n\in H^n$ corresponding them by realification, are linearly independent in H^n.

Note 1. Any system that has strong linear independent real vectors is of course linear independent, but the converse is not always true.

For example, the vectors $\vec{x}=(1,0,...,0)$, $\vec{y}=(0,1,0,...,0)\in V$ are linearly independent, but not strongly linearly independent, because with respect to the action (5) the vectors $\vec{x}$ and $\vec{y}$ corresponding to the quaternion vectors $a=(1,0,...,0)$ and $b=(i,0,0,...,0)$, respectively. The vectors a and b are not linearly independent in the quaternion space H^n.

The group of real representations of the symplectic group Sp (n)

The realification operation described above is a one-to-one correspondence between the spaces H^n and V. It follows that every element $\vartheta\in GL\,(H^n)$ defines a linear transformation $\vartheta'\in GL\,(V)$. Also, $GL\,(H^n)$ can be regarded as a subgroup in $GL\,(V)$ using the isomorphism in $\vartheta\leftrightarrow\vartheta'$. Then $Sp\,(n)$ can be regarded also as a subgroup of $GL\,(V)$. This subgroup is called *the real representation* of $Sp\,(n)$, [16]. We define this group as follows.

Let $\langle x,y\rangle$ be defined in the form (2). We denote the coefficients in front of its units 1, i, j, k by $\Omega_1\,(\vec{x},\vec{y})$, $\Omega_i\,(\vec{x},\vec{y})$, $\Omega_j\,(\vec{x},\vec{y})$, $\Omega_k\,(\vec{x},\vec{y})$, respectively, i.e.,

$$\langle x,y\rangle=\Omega_1\,(\vec{x},\vec{y})-\Omega_i\,(\vec{x},\vec{y})\,i-\Omega_j\,(\vec{x},\vec{y})\,j-\Omega_k\,(\vec{x},\vec{y})\,k, \tag{6}$$

where $x,y\in H^n$, $\vec{x},\vec{y}\in V$, also

1. $\Omega_1\,(\vec{x},\vec{y})=\sum\limits_{l=1}^{n}(x_{l1}y_{l1}+x_{l2}y_{l2}+x_{l3}y_{l3}+x_{l4}y_{l4})$;
2. $\Omega_i\,(\vec{x},\vec{y})=\sum\limits_{l=1}^{n}(x_{l1}y_{l2}-x_{l2}y_{l1}+x_{l3}y_{l4}-x_{l4}y_{l3})$;
3. $\Omega_j\,(\vec{x},\vec{y})=\sum\limits_{l=1}^{n}(x_{l1}y_{l3}-x_{l3}y_{l1}+x_{l4}y_{l2}-x_{l2}y_{l4})$;
4. $\Omega_k\,(\vec{x},\vec{y})=\sum\limits_{l=1}^{n}(x_{l1}y_{l4}-x_{l4}y_{l1}+x_{l2}y_{l3}-x_{l3}y_{l2})$.

Obviously, a symplectic transformation leaves invariant of the bilinear form $\langle x,y\rangle$. Then, the corresponding real transformation to it leaves invariant the bilinear forms $\Omega_1,\Omega_i,\,\Omega_j,\Omega_k$, (see,[16]). We will have the following definition from this property.

Definition 2. The group of linear transformations $\vartheta'\in GL\,(V)$ is called *a group of the real representation* $Sp(n)$ if it satisfies the following conditions:

$$\left\{\begin{array}{c} \vartheta'\in GL\,(V):\,\Omega_1\,(\vartheta'\vec{x},\vartheta'\vec{y})=\Omega_1\,(\vec{x},\vec{y})\,,\ \Omega_i\,(\vartheta'\vec{x},\vartheta'\vec{y})=\Omega_i\,(\vec{x},\vec{y})\,, \\ \Omega_j\,(\vartheta'\vec{x},\vartheta'\vec{y})=\Omega_j\,(\vec{x},\vec{y})\,,\ \Omega_k\,(\vartheta'\vec{x},\vartheta'\vec{y})=\Omega_k\,(\vec{x},\vec{y}) \end{array}\right\}.$$

It is known that each transformation $\vartheta' \in GL(V)$ can be uniquely represented by the matrix $g \in GL(4n, R)$. This allows us to define the group of real representations of $Sp(n)$ using matrices $g \in GL(4n, R)$. To do this, we use Definition 2 and the following:

$$\Omega_1(\vec{x},\vec{y}) = \vec{x}(\vec{y})^T, \Omega_i(\vec{x},\vec{y}) = \vec{x}I(\vec{y})^T,\ \Omega_j(\vec{x},\vec{y}) = \vec{x}J(\vec{y})^T,\ \Omega_j(\vec{x},\vec{y}) = \vec{x}K(\vec{y})^T, \tag{7}$$

where

$$I = \begin{bmatrix} I_1 & \theta & ... & \theta \\ \theta & I_1 & ... & \theta \\ \vdots & \vdots & ... & \vdots \\ \theta & \theta & ... & I_1 \end{bmatrix},\ J = \begin{bmatrix} J_1 & \theta & ... & \theta \\ \theta & J_1 & ... & \theta \\ \vdots & \vdots & ... & \vdots \\ \theta & \theta & ... & J_1 \end{bmatrix},\ K = \begin{bmatrix} K_1 & \theta & ... & \theta \\ \theta & K_1 & ... & \theta \\ \vdots & \vdots & ... & \vdots \\ \theta & \theta & ... & K_1 \end{bmatrix};$$

here $\theta-$ is 4th ordered zero matrix, also

$$I_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},\ J_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},\ K_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

As a result, we have the following definition of the real representation $Sp(n)$ given by the matrices $g \in GL(4n, R)$:

Definition 3. The group of matrix $g \in GL(4n, R)$ is said to be the real representation $Sp(n)$, if it satisfies the following conditions:

$$\left\{g \in GL(4n, R):\ gg^T = E,\ gIg^T = I,\ gJg^T = J,\ gKg^T = K,\ \det g = 1\right\}, \tag{8}$$

where E is 4n th ordered unit matrix.

In what follows, we consider only the group of real representation of $Sp(n)$, and denote it by $\mathfrak{Sp}(4n)$.

It is known that group of matrices $g \in GL(n, R)$ that simultaneously satisfy the conditions $gg^T = E$, $gIg^T = I$, $\det g = 1$ is called orthogonal-symplectic group, and denoted by $O(n, R) \cap Sp(n, R)$, (see, [19]). According to this definition, we can also write the group $\mathfrak{Sp}(4n)$ in the following form

$$\mathfrak{Sp}(4n) = O(4n, R) \cap Sp(4n, R) \cap G_1 \cap G_2,$$

where the groups G_1 and G_2 are defined in form as follows, respectively

$$G_1 = \left\{g \in GL(4n, R) : gJg^T = J\right\},\ G_2 = \left\{g \in GL(4n, R) : gKg^T = K\right\}.$$

Lemma 1. *Let groups $\mathfrak{Sp}(4n)$, G_1 and G_2 be given. Then the relations*

$$G_1 \cong Sp(4n, R), G_2 \cong Sp(4n, R), G_1 \cong G_2$$

hold under the mappings $\varphi_1(g_1) = A_1 g_1 A_1$, $\varphi_2(g_2) = A_2 g_2 A_2$ and $\varphi_3(g_2) = A_3 g_2 A_3$ respectively, where $g_1, g_3 \in G_1$, $g_2, \varphi_3(g_3) \in G_2$, $\varphi_1(g_1), \varphi_2(g_2) \in Sp(4n, R)$,

$$A_1 = \begin{bmatrix} A_{11} & \theta & ... & \theta \\ \theta & A_{11} & ... & \theta \\ \vdots & \vdots & \vdots & \vdots \\ \theta & \theta & ... & A_{11} \end{bmatrix}, A_2 = \begin{bmatrix} A_{21} & \theta & ... & \theta \\ \theta & A_{21} & ... & \theta \\ \vdots & \vdots & \vdots & \vdots \\ \theta & \theta & ... & A_{21} \end{bmatrix}, A_3 = \begin{bmatrix} A_{31} & \theta & ... & \theta \\ \theta & A_{31} & ... & \theta \\ \vdots & \vdots & \vdots & \vdots \\ \theta & \theta & ... & A_{31} \end{bmatrix};$$

also,

$$A_{11} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, A_{21} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, A_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

as well as, for the matrices A_1, A_2, A_3 the following equalities are hold:

a) $A_1 J A_1 = I,\ A_1^2 = E$; b) $A_2^T = A_2,\ A_2 J A_2 = I,\ A_2^2 = E$; c) $A_3^2 = E,\ A_3 J A_3 = K,\ A_3^T = A_3$.

Proof. To prove Lemma 1, it suffices to show that the relation $G_1 \cong Sp(4n, R)$ holds with respect to the mapping $\varphi_1(g_1) = A_1 g_1 A_1$. Because the others can be proved in the same way. Let a mapping $\varphi : G_1 \to Sp(4n, R)$ be of the form $\varphi(g_1) = A_1 g_1 A_1$.

1. *The mapping φ_1 of the group G_1 into the group $Sp(n,R)$ is a homomorphism of G_1 into $Sp(n,R)$.* Indeed, for any $g_1^{'},\ g_1^{''} \in G_1$

$$\varphi\left(g_1^{'}g_1^{''}\right) = A_1\left(g_1^{'}g_1^{''}\right)A_1 = A_1g_1^{'}Eg_1^{''}A_1 = A_1g_1^{'}A_1^2g_1^{''}A_1 = \varphi\left(g_1^{''}\right)\varphi\left(g_1^{''}\right),$$

i.e., the equality $\varphi\left(g_1^{'}g_1^{''}\right) = \varphi\left(g_1^{''}\right)\varphi\left(g_1^{''}\right)$ is hold;

2. *For the φ_1 homomorphism the relation $ker\varphi_1 = \{e_1\}$ is hold, where $ker\varphi_1$ is kernel of the homomorphism φ_1, i.e., $ker\varphi_1 = {\varphi_1}^{-1}(e_1)$, also e_1 is the unit element of group $Sp(n,R)$.*

It will be shown that the above claim is true as follows:

let be ${\varphi_1}^{-1}(e_1) = G'$. Then the equality $\varphi_1(G') = e_1$ holds. From here follows the equality $\varphi_1({g_1}') = e_1 = A_1{g_1}'A_1$ for any ${g_1}' \in G'$. By multiplying this equality by A_1 on both from the left and right, we have ${g_1}' = e_1$. From the fact that the last equality is valid for all ${g_1}' \in G'$, the relation $G' = \{e_1\}$ follows. This implies that the equality $ker\varphi_1 = \{e_1\}$ is true.

Statements 1 and 2 above show that the mapping φ_1 will be an isomorphic mapping of the group G_1 into the group $Sp(n,R)$. Lemma 1 has been proved. □.

From Lemma 1, we will have the corollary in the following.

Corollary 1. $\mathfrak{Sp}\,(4n) \cong O\,(4n,R) \cap Sp\,(4n,R)$.

The ring of invariant polynomials with respect to the action of the group $\mathfrak{Sp}\,(4n)$

In this section, we study the ring of $\mathfrak{Sp}(4n)-$invariant polynomials and the problem of constructing a system of its generators. In addition, we define the relationship between them.

Let V be a $4n$ dimensional vector space, which the realification of H^n. The elements of the space V will be represented as $4n$ dimensional row-vector, and denoted by $\vec{x}$. Let be $G \subset GL(4n,R)$. As an action of the group G to the space V is defined as right multiplication of the matrix $g \in G$ to the row-vector $\vec{x} \in V$, i.e., $(g,\vec{x}) = \vec{x}g$. Let $R\left[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}\right]$ be the ring of real polynomials in $4n$ vector arguments, where $\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n} \in V$.

Definition 4. The polynomial function $f\left[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}\right]$ is called $G-$ **invariant**, if the equality

$$f\left[\vec{x}_1g,\vec{x}_2g,...,\vec{x}_{4n}g\right] = f\left[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}\right]$$

is true for all $g \in G$.

We denote the set of all invariant polynomials with $R[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}]^G$. It is plain, this set is a subring of the ring $R[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}]^G$ with respect to operations defined on the ring $R\left[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}\right]$, i.e.,

$$R[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}]^G \subset R\left[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}\right].$$

Let the set $\Sigma = \{\varepsilon_l\}_{l\in\Delta}$ is consisted from elements of $R\left[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}\right]$, where Δ is a set in finite number of natural numbers.

Definition 5. The elements $\varepsilon_1,\ \varepsilon_2,...,\varepsilon_s \in \Sigma$ are called *algebraical dependent*, if that exists such polynomial $P\left[\vec{x}_1,\vec{x}_2,...,\vec{x}_s\right]$ in $R\left[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}\right]$, then be $P\left(\varepsilon_1,\ \varepsilon_2,...,\varepsilon_s\right) = 0$, otherwise, these elements are called *algebraical independent*, over $R[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}]^G$.

The set $\Sigma = \{\varepsilon_l\}_{l\in\Delta}$ is called *generating system* of the ring $R[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}]^G$, if an arbitrary element $f\left[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}\right] \in R[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}]^G$ can be generating by applying a finite number of operations of the ring $R[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}]^G$ to the elements of $\Sigma = \{\varepsilon_l\}_{l\in\Delta}$. A system of algebraical independent generators is called *a integrity basis* of the ring $R[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}]^G$. In the following, we consider ***the problem of describing a system of generators in*** $R[\vec{x}_1,\vec{x}_2,...,\vec{x}_{4n}]^G$ ***for*** $G = \mathfrak{Sp}\,(4n)$.

Let there be given several $G-$invariant polynomials of some vector arguments $u_1,\ u_2,...$, i.e.,

$$\varphi_1\left[u_1,\ u_2,...\right],\ \varphi_2\left[u_1,\ u_2,...\right],.... \tag{9}$$

The system (9) will be a complete table of typical basic invariants for m arguments, if it changes into an integrity basis for invariants of m arguments $\vec{x_1},\vec{x_2},...,\vec{x_m}$, by substituting for $u_1,\ u_2,...$ these arguments in all possible combinations (repetitions included). Also, for the table of typical basic invariants of a linear group of n-th degree to be complete with respect to any m argument, it is sufficient that it is true for n arguments, (see, [6]). Using these facts, we prove the following theorem.

Theorem 1. *Let G be group $\mathfrak{Sp}\,(4n)$. Any $\mathfrak{Sp}(4n)-$invariant polynomial is expressed an integrally rational manner by the elements of the system*

$$\Omega_\alpha\left(\vec{x}_l, \vec{\xi}_m\right),\ \alpha \in \{1,\, i,\, j,\, k\} \tag{10}$$

where $\vec{x}_l \in V$, $\vec{\xi}_m \in V^*$, V^* is the adjoint space for the space V.

Proof. To prove Theorem 1, it is sufficient to show that, according to the above facts, the statement of the theorem is true for the sets of vectors $\vec{x}_1, \vec{x}_2, ..., \vec{x}_{4n} \in V$ and $\vec{\xi}_1,\, \vec{\xi}_2, ..., \vec{\xi}_{4n} \in V^*$. In other words it is sufficient to show that arbitrary $\mathfrak{Sp}\,(4n)-$ invariant polynomial of these vector arguments, be generated through forms (10).

Obviously, we can express any polynomial $p\left[\vec{x}_1, \vec{x}_2, ... \left|\vec{\xi}_1,\, \vec{\xi}_2, ...\right.\right]$ by the form $P\left\{\left\langle \vec{x}_l \left|\vec{\xi}_m\right.\right\rangle\right\}$, where $\left\langle \vec{x}_l \left|\vec{\xi}_m\right.\right\rangle = \vec{x}_l\vec{\xi}_m = \sum\limits_{n=1}^{4n} x_{ln}\xi_{mn}$.

Let $p\left[\vec{x}_1, \vec{x}_2, ..., \vec{x}_{4n} \left|\vec{\xi}_1,\, \vec{\xi}_2, ..., \vec{\xi}_{4n}\right.\right]$ be any $\mathfrak{Sp}\,(4n)\,-$invariant polynomial. Using the transformation $g \in \mathfrak{Sp}\,(4n)$, we can pass the vector arguments $\vec{x}_1, \vec{x}_2, ..., \vec{x}_{4n} \in V$ to the vectors $\vec{e}_1, \vec{e}_2, ..., \vec{e}_{4n}$, which the standard basis of vectors in V.

Let $g \in \mathfrak{Sp}\,(4n)$. Then we have the system of equations $\vec{x}_l g = \vec{e}_l,\ \left(l = \overline{1, 4n}\right)$. Also, considering that $\vec{e}_{4t+2} = \vec{e}_{4t+1}I,\ \vec{e}_{4t+3} = \vec{e}_{4t+1}J,\ \vec{e}_{4t+4} = \vec{e}_{4t+1}K,\ (t = \overline{0, n-1})$ and $g \in \mathfrak{Sp}\,(4n)$, we get the system of equations

$$\vec{x}_{4t+2} = \vec{x}_{4t+1}I,\ \vec{x}_{4t+3} = \vec{x}_{4t+1}J,\ \vec{x}_{4t+4} = \vec{x}_{4t+1}K. \tag{11}$$

In this case, the coordinates of each vector of the set vectors $\{\vec{x}_l,\, \vec{x}_{l+1},\, \vec{x}_{l+2},\, \vec{x}_{l+3}\}$ are determined by the coordinates of the vector $\vec{x}_l$, where $l = 4t+1,\ t = \overline{0, n-1}$. In what follows, we denote by X the matrix in the form $(x_{lm})_{l,m=1}^{4n}$. Then, the matrix X with the help of equations (10) is defined in the following form

$$X = \begin{bmatrix} X_{11} & X_{15} & ... & X_{1(4n-3)} \\ X_{51} & X_{55} & ... & X_{5(4n-3)} \\ \vdots & \vdots & ... & \vdots \\ X_{(4n-3)1} & X_{(4n-3)5} & ... & X_{(4n-3)(4n-3)} \end{bmatrix},$$

where X_{lm} will be as follows:

$$X_{lm} = \begin{bmatrix} x_{lm} & x_{lm+1} & x_{lm+2} & x_{lm+3} \\ -x_{lm+1} & x_{lm} & -x_{lm+3} & x_{lm+2} \\ -x_{lm+2} & x_{lm+3} & x_{lm} & -x_{lm+1} \\ -x_{lm+3} & -x_{lm+2} & x_{lm+1} & x_{lm} \end{bmatrix}.$$

In addition, we have the equality $g = X^{-1}$ from the equation $Xg = E$. Also, we obtain to the equality $g = X^T$ from $g \in \mathfrak{Sp}(4n)$. In turn, during the transformation the set $\{\vec{x}_1,\, \vec{x}_2, ..., \vec{x}_{4n}\}$ to the set $\{\vec{e}_1,\, \vec{e}_2, ..., \vec{e}_{4n}\}$ respectively the vectors $\vec{\xi}_1, \vec{\xi}_2, ..., \vec{\xi}_{4n}$ will changed into the system of vectors $\vec{\xi}_1^{'}, \vec{\xi}_2^{'}, ..., \vec{\xi}_{4n}^{'}$ and defined them as follows:

we denote the matrices $(\xi_{lm})_{l,m=1}^{4n}$ and $\left(\xi_{lm}^{'}\right)_{l,m=1}^{4n}$ respectively by Ξ and Ξ'. Then, we have an equation as follows $\Xi^{'} = g^T\Xi = X\Xi$. From this equation it follows that $\xi_{lm}^{'} = \Omega_\alpha\left(\vec{x}_s, \vec{\xi}_m\right)$, where $l, m = \overline{1, 4n}$,

$$\alpha = \begin{cases} 1,\ if\ l = 4s-3,\ then\,be\ s = \overline{1, n}; \\ i,\ if\ l = 4s-2,\ then\,be\ s = \overline{1, n}; \\ j,\ if\ l = 4s-1,\ then\,be\ s = \overline{1, n}; \\ k,\ if\ l = 4s,\ then\,be\ s = \overline{1, n}. \end{cases} \tag{12}$$

Then, we can see that each vector $\vec{\xi}_l^{'}$ is defined as an algebraic expression of the bilinear form $\Omega_\alpha\left(\vec{x}_s, \vec{x}_m\right)$. In addition, the equation

$$p\left[\vec{x}_1, ..., \vec{x}_{4n} \left|\vec{\xi}_1, ..., \vec{\xi}_{4n}\right.\right] = p\left[\vec{e}_1, ..., \vec{e}_{4n} \left|\vec{\xi}_1^{'}, ..., \vec{\xi}_{4n}^{'}\right.\right] = P\left[\xi_{lm}^{'}\right]$$

is true for any $\mathfrak{Sp}(4n)-$invariant polynomial $p\left[\vec{x}_1,...,\vec{x}_{4n}\middle|\vec{\xi}_1,...,\vec{\xi}_{4n}\right]$. Then it follows that any $\mathfrak{Sp}(4n)-$invariant polynomial $p\left[\vec{x}_1,...,\vec{x}_{4n}\middle|\vec{\xi}_1,...,\vec{\xi}_{4n}\right]$ algebraically expressed by the forms $\Omega_\alpha(\vec{x}_s,\vec{x}_m)$. Theorem 1 has been proved. □.

It is known that the second important problem in the course of invariant theory is *the determination of the relationship between typical basic invariants.* Accordingly, below we define the relationship between the elements of the system (10).

Let the scalar product in the space H^n be given by the form $\langle\,,\,\rangle$. In this case, consider the product

$$\langle x_1,\,x_2\rangle\langle x_3,\,x_4\rangle...\langle x_{2n-1},\,x_{2n}\rangle\,, \tag{13}$$

corresponding to the vectors $x_1,\,x_2,x_3,...,x_{2n}\in H^n$, where $\vec{x}_l\neq\theta$, $l=\overline{1,2n}$. Replacing each $\langle\,,\,\rangle$ bilinear form given in the product (13) by (6), we obtain a formula of the following form

$$\langle x_1,x_2\rangle...\langle x_{2n-1},x_{2n}\rangle = F^{\alpha_1,...,\alpha_n}+F^{\beta_1,...,\beta_n}i+F^{\gamma_1,...,\gamma_n}j+F^{\nu_1,...,\nu_n}k, \tag{14}$$

where

$$F^{\alpha_1,...,\alpha_n}=\sum c_\alpha\Omega_{\alpha_1}(\vec{x}_1,\,\vec{x}_2)...\Omega_{\alpha_n}(\vec{x}_{2n-1},\,\vec{x}_{2n}),\quad \alpha_1\cdot\alpha_2\cdot...\cdot\alpha_n=\pm1;$$

$$F^{\beta_1,...,\beta_n}=\sum c_\beta\Omega_{\beta_1}(\vec{x}_1,\,\vec{x}_2)...\Omega_{\beta_n}(\vec{x}_{2n-1},\,\vec{x}_{2n}),\quad \beta_1\cdot\beta_2\cdot...\cdot\beta_n=\pm i;$$

$$F^{\gamma_1,...,\gamma_n}=\sum c_\gamma\Omega_{\gamma_1}(\vec{x}_1,\,\vec{x}_2)...\Omega_{\gamma_n}(\vec{x}_{2n-1},\,\vec{x}_{2n}),\quad \gamma_1\cdot\gamma_2\cdot...\cdot\gamma_n=\pm j;$$

$$F^{\nu_1,...,\nu_n}=\sum c_\nu\Omega_{\nu_1}(\vec{x}_1,\,\vec{x}_2)...\Omega_{\nu_n}(\vec{x}_{2n-1},\,\vec{x}_{2n}),\quad \nu_1\cdot\nu_2\cdot...\cdot\nu_n=\pm k;$$

$$c_\alpha=(-1)^q sign(\alpha_1\cdot\alpha_2\cdot...\cdot\alpha_n)\,,\quad c_\beta=(-1)^q sign(\beta_1\cdot\beta_2\cdot...\cdot\beta_n)\,;$$

$$c_\gamma=(-1)^q sign(\gamma_1\cdot\gamma_2\cdot...\cdot\gamma_n)\,,\quad c_\nu=(-1)^q sign(\nu_1\cdot\nu_2\cdot...\cdot\nu_n)\,;$$

$\alpha_\ell,\,\beta_\ell,\,\gamma_\ell,\nu_\ell\in\{1,\,i,\,j,\,k\}$, $q-$ the number of imaginary units in the product $\{\omega_1\cdot\omega_2\cdot...\cdot\omega_n\}$, $\omega_\ell\in\{\alpha_\ell,\,\beta_\ell,\,\gamma_\ell,\,\nu_\ell\}$, $\ell=\overline{1,n}$.

For example (see, [22]), $\langle x_1,x_2\rangle\langle x_3,x_4\rangle=$

$$=[\Omega_1(x_1,x_2)\Omega_1(x_3,x_4)-\Omega_i(x_1,x_2)\Omega_i(x_3,x_4)-\Omega_j(x_1,x_2)\Omega_j(x_3,x_4)-\Omega_k(x_1,x_2)\Omega_k(x_3,x_4)]+$$

$$+i[-\Omega_1(x_1,x_2)\Omega_i(x_3,x_4)-\Omega_i(x_1,x_2)\Omega_1(x_3,x_4)+\Omega_j(x_1,x_2)\Omega_k(x_3,x_4)-\Omega_k(x_1,x_2)\Omega_j(x_3,x_4)]+$$

$$+j[-\Omega_1(x_1,x_2)\Omega_j(x_3,x_4)-\Omega_j(x_1,x_2)\Omega_1(x_3,x_4)+\Omega_k(x_1,x_2)\Omega_i(x_3,x_4)-\Omega_i(x_1,x_2)\Omega_k(x_3,x_4)]+$$

$$+k[-\Omega_1(x_1,x_2)\Omega_k(x_3,x_4)-\Omega_k(x_1,x_2)\Omega_1(x_3,x_4)+\Omega_i(x_1,x_2)\Omega_j(x_3,x_4)-\Omega_j(x_1,x_2)\Omega_i(x_3,x_4)]=$$

$$=F^{\alpha_1,\alpha_2}(x_1,x_2,x_3,x_4)+F^{\beta_1,\beta_2}(x_1,x_2,x_3,x_4)\,i+F^{\gamma_1,\gamma_2}(x_1,x_2,x_3,x_4)\,j+F^{\nu_1,\nu_2}(x_1,x_2,x_3,x_4)\,.$$

Using the above definition and formulas, we will defined the relationship between the elements of system (10). To do this, we use an n linear independent vectors $x_1,...,x_n\in H^n$ and the properties of the determinant of the Gram matrix composed of them. Further, let us denote the Gram matrix as $\Gamma(x_1,...,x_n)$. It is known that the Gram matrix $\Gamma(x_1,...,x_n)$ is a Hermitian quaternion matrix of order n and it's determinant defines with formula in the following

$$\det\nolimits_l\Gamma(x_1,...,x_n)=\sum_{\sigma\in S_n}(-1)^{n-\kappa}\left\langle x_l,x_{l_{m_1}}\right\rangle\left\langle x_{l_{m_1}},x_{l_{m_1+1}}\right\rangle...\left\langle x_{l_{m_1+\delta_1}},x_l\right\rangle...\left\langle x_{l_{m_\kappa}},x_{l_{m_\kappa+1}}\right\rangle...\left\langle x_{l_{m_\kappa+\delta_\kappa}},x_n\right\rangle, \tag{15}$$

where S_n is a group of the permutation of the set $\{1,2,...,n\}$, l is number of rows, $\sigma=(l,l_{m_1},l_{m_1+1},...,l_{m_1+\delta_1})...(l_{m_\kappa},l_{m_\kappa+1},...,l_{m_\kappa+\delta_\kappa})\in S_n$, $\kappa-$ is number of cyclic; In addition, the following properties hold for the Gram determinant:

Proposition 1. *If the vectors* $\{x_1,x_2,...,x_n\}$ *is linearly independent in* H^n*, then the relations* $\det_l\Gamma(x_1,...,x_n)\in R$ *and* $\det_l\Gamma(x_1,...,x_n)\neq0$ *hold, where* l *is the number of rows;*

Proposition 2. *Let* $\{x_1,...,x_n,x\}$ *a set of the first* n *linearly independent vectors in the space* H^n*. Then for the Gram matrix consisting of them the equality* $\det_l\Gamma(x_1,...,x_n,x)=0$ *is true, where* $x=\lambda_1x_1+\lambda_2x_2+...+\lambda_nx_n$;

Proposition 3. *Let $\{x_1, ..., x_{n+1}, x_{n+2}\}$ a set of the first n linearly independent vectors in the space H^n. Then, the minor determinant $M_{n+2,1}$ of order $(n+1)-th$ of $\det_l \Gamma(x_1, ..., x_n, x_{n+1}, x_{n+2})$ is equal zero, i.e.,*

$$M_{n+2,1} = \sum_{\sigma_1' \in \bar{S}_{n+1}} (-1)^{n+1-\kappa} \langle x_1, x_{l_{m_1}} \rangle \langle x_{l_{m_1}}, x_{l_{m_1+1}} \rangle ... \langle \widetilde{x_{l_{m_1+\delta_1}}, x_1} \rangle ... \langle x_{l_{m_\kappa}}, x_{l_{m_\kappa+1}} \rangle ... \langle x_{l_{m_\kappa+\delta_\kappa}}, x_{l_{m_\kappa}} \rangle = 0$$

where the permutation σ_1' and the set $\bar{S}_{n+1}$ are obtained as follows:

let $\sigma' \in S_{n+2}$ be a permutation in form as the follows

$$\sigma' = (n+2\ 1\ l_{m_1}\ ...\ l_{m_1+\delta_1}) ... (l_{m_\kappa}\ l_{m_\kappa+1}\ ...\ l_{m_\kappa+\delta_\kappa});$$

denote by σ_1' that the decomposition

$$(1\ l_{m_1}\ ...\ l_{m_1+\delta_1}) ... (l_{m_\kappa}\ l_{m_\kappa+1}\ ...\ l_{m_\kappa+\delta_\kappa})$$

Also, we denote by $\overline{S}_{n+1}$ that the set of all the decomposition σ_1'; $\langle \widetilde{x_{l_{m_1+\delta_1}}, x_1} \rangle = \langle x_{l_{m_1+\delta_1}}, x_{n+2} \rangle$.

Proposition 1-3 follows from the properties of Hermitian quaternion matrices, (see, [21]).

Now, we define the relations corresponding to Propositions 1-3 for the set of strongly linearly independent vectors $\vec{x}_1, ..., \vec{x}_n$ given in space V.

Let a set $B_1 = \{1, 2, ..., n\}$ and a permutation group S_n, consisting of elements of the set B_1 be given. It is known that any permutation

$$\begin{pmatrix} 1 & ... & l_{m_1} & ... & l_{m_1+\delta_1} & l_{m_2} & ... & l_{m_2+\delta_2} & ... & l_{m_\kappa} & ... & n \\ l_{m_1} & ... & l_{m_1+1} & ... & 1 & l_{m_2+1} & ... & l_{m_2} & ... & l_{m_\kappa+1} & ... & l_{m_\kappa} \end{pmatrix} \in S_n$$

can be represented as a decomposition

$$(1, l_{m_1}, ..., l_{m_1+\delta_1})(l_{m_2}, l_{m_2+1}, ..., l_{m_2+\delta_2}) ... (l_{m_\kappa}, l_{m_\kappa+1}, ..., n) \tag{16}$$

where $l_{m_s+\delta_s} = \overline{1,n}$, $\delta_s \in Z_0^+$, $s = \overline{1,\kappa}$, $l_{m_2} < l_{m_3} < ... < l_{m_\kappa}$, κ is number of cyclic. We denote by υ the set of ordered pairs corresponding to decomposition (16), i.e.,

$$\{(1, l_{m_1}), ..., (l_{m_1+\delta_1}, 1), (l_{m_2}, l_{m_2+1}), ..., (l_{m_2+\delta_2}, l_{m_2}), ..., (l_{m_\kappa}, l_{m_\kappa+1}), ..., (n, l_{m_\kappa})\},$$

and denote by ρ a bijective mapping from B_1 to υ. We also denote the set of all mappings ρ by A_ρ and by $F_{\rho_\tau}^{\alpha_1,...,\alpha_n}$ the product

$$\Omega_{\alpha_1}\left(\vec{x}_{l_{m_{s_1}}}, \vec{x}_{l'_{m_{s_1}}}\right) \Omega_{\alpha_2}\left(\vec{x}_{l_{m_{s_2}}}, \vec{x}_{l'_{m_{s_2}}}\right) \cdot ... \cdot \Omega_{\alpha_n}\left(\vec{x}_{l_{m_{s_n}}}, \vec{x}_{l'_{m_{s_n}}}\right),$$

where $\alpha_1, ..., \alpha_n \in \{1, i, j, k\}$, $\left\{l_{m_s}, l'_{m_s}\right\} = \rho_\tau^{-1}(m_s)$, $l_{m_s} < l'_{m_s}$, $m_s \in B_1$, $\tau = \overline{1, n!}$.

Lemma 2. *Let $\{\vec{x}_1, \vec{x}_2, ..., \vec{x}_n\}$ be a set of strong linearly independent vectors in V. If the condition $\alpha_1\alpha_2 \cdot ... \cdot \alpha_n = \pm 1$ holds, then the relation*

$$F(\vec{x}_1, \vec{x}_2, ..., \vec{x}_n) = \sum_{\rho_\tau \in A_\rho} (-1)^{n-\kappa} c_{\rho_\tau}^{\alpha} F_{\rho_\tau}^{\alpha_1\alpha_2 \cdot ... \cdot \alpha_n} \neq 0 \tag{17}$$

is valid for the set of vectors $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ where κ is number of the permutation σ_τ $(\upsilon_\tau \leftrightarrow \sigma_\tau)$, also

$$c_{\bar{\rho}_\tau}^{\alpha} = (-1)^q sign\{\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n\} \times$$

$$\times sign\left\{\Omega_{\alpha_1}\left(\vec{x}_{l_{m_{s_1}}}, \vec{x}_{l'_{m_{s_1}}}\right) \Omega_{\alpha_2}\left(\vec{x}_{l_{m_{s_2}}}, \vec{x}_{l'_{m_{s_2}}}\right) \cdot ... \cdot \Omega_{\alpha_2}\left(\vec{x}_{l_{m_{s_n}}}, \vec{x}_{l'_{m_{s_n}}}\right)\right\},$$

$q-$ is number of imaginary units in the product $\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n$.

Proof. To prove Lemma 2, we use Proposition 1 and equality (14). According to Proposition 1, the relations $\det_l \Gamma(x_1, x_2, ..., x_n) \in R$ and $\det_l \Gamma(x_1, x_2, ..., x_n) \neq 0$ are valid for the linearly independent vectors $x_1, x_2, ..., x_n \in H^n$; from equality (15) we have that, for $l = 1$, the formula

$$\det_1 \Gamma(x_1, ..., x_n) = \sum_{\sigma \in S_n} (-1)^{n-\kappa} \langle x_1, x_{l_{m_1}} \rangle \langle x_{l_{m_1}}, x_{l_{m_1+1}} \rangle ... \left\langle x_{l_{m_\kappa+\delta_\kappa}}, x_n \right\rangle. \tag{18}$$

Taking into account the definition of the mapping ρ and the equality $\Omega_1\left(x_{l_{m_\kappa}}, x_{l'_{m_\kappa}}\right) = \Omega_1\left(x_{l'_{m_\kappa}}, x_{l_{m_\kappa}}\right)$, $\Omega_{\alpha_1}\left(x_{l_{m_\kappa}}, x_{l'_{m_\kappa}}\right) = -\Omega_{\alpha_1}\left(x_{l'_{m_\kappa}}, x_{l_{m_\kappa}}\right)$, $(\alpha_1 \in \{i,j,k\})$ we obtain the following relation by applying equality (14) to the right of formula (18):

$$\det{}_1\Gamma(x_1,...,x_n) = \sum_{\rho_\tau \in A_\rho} (-1)^{n-\kappa} c^{\alpha}_{\rho_\tau} F^{\alpha_1\cdot...\cdot\alpha_n}_{\rho_\tau} + i \sum_{\rho_\tau \in A_\rho} (-1)^{n-\kappa} c^{\beta}_{\rho_\tau} F^{\beta_1\cdot...\cdot\beta_n}_{\rho_\tau} + \\ + j \sum_{\rho_\tau \in A_\rho} (-1)^{n-\kappa} c^{\gamma}_{\rho_\tau} F^{\gamma_1\cdot...\cdot\gamma_n}_{\rho_\tau} + k \sum_{\rho_\tau \in A_\rho} (-1)^{n-\kappa} c^{\nu}_{\rho_\tau} F^{\nu_1\cdot...\cdot\nu_n}_{\rho_\tau}, \quad (19)$$

where the vectors $\vec{x}_1,...,\vec{x}_n$ corresponds to the vectors $x_1,...,x_n$ with respect to the action of realification.

In this case, since the vectors $x_1,...,x_n$ are linearly independent, it is clear that the vectors $\vec{x}_1,...,\vec{x}_n$ are strongly linearly independent; Since $\det_1\Gamma(x_1,...,x_n) \in R$, in equality (19) the coefficients in front of the imaginary units i,j,k are equal to zero; From $\det_1\Gamma(x_1,...,x_n) \neq 0$ it follows that statements of Lemma 2. □

Lemma 3. *Let $\{\vec{x}_l\}_{l=1}^{n+1}$ be a sequence of vectors $\vec{x}_1,\vec{x}_2,...,\vec{x}_n,\vec{x}_{n+1}$, the firs n of which are strongly linearly independent. If the condition $\alpha_1\alpha_2\cdot...\cdot\alpha_{n+1} = \pm 1$ holds, then the relation*

$$F(\vec{x}_1,\vec{x}_2,...,\vec{x}_{n+1}) = \sum_{\rho_\tau \in A_\rho} (-1)^{n+1-\kappa} c^{\alpha}_{\rho_\tau} F^{\alpha_1\alpha_2\cdot...\cdot\alpha_{n+1}}_{\rho_\tau} = 0 \quad (20)$$

is valid for the sequence $\{\vec{x}_l\}_{l=1}^{n+1}$, where κ is a number of cycles in permutation σ, which the corresponding to the mapping ρ_τ, also

$$c^{\alpha}_{\rho_\tau} = (-1)^q sign\{\alpha_1\cdot...\cdot\alpha_{n+1}\} \cdot sign\left\{\Omega_{\alpha_1}\left(x_{l_{m_1}}, x_{l'_{m_1}}\right)\cdot...\cdot\Omega_{\alpha_{n+1}}\left(x_{l_{m_{n+1}}}, x_{l'_{m_{n+1}}}\right)\right\}.$$

The statement of Lemma 3 follows from Property 2, equalities (14) and (18). Only in this case is considered the permutation of $\sigma \in S_{n+1}$, the corresponding set υ that it, and the set of mapping $\rho : \{1,2,...,n+1\} \to \upsilon$.

Let us now define the relation that follows from Proposition 3.

Let a set $B_2 = \{1,2,...,n+1,n+2\}$ and a group S_{n+2} be the permutation group consisting of the elements of B_2 be given. We obtain the elements of the group S_{n+2} in the following form

$$\sigma = \begin{pmatrix} n+2 & 1 & 2 & ... & n+1 \\ 1 & l_{m_{s_1}} & l_{m_{s_2}} & ... & l_{m_{s_{n+1}}} \end{pmatrix}.$$

Also, we can be represented of this permutations in form decomposition of independent cycles, i.e.,

$$\sigma = (n+2, 1, l_{m_1},...,l_{m_1+\delta_1})(l_{m_2}, l_{m_2+1},...,l_{m_2+\delta_2})...(l_{m_\kappa}, l_{m_\kappa+1},...,l_{m_\kappa+\delta_\kappa}), \quad (21)$$

where $l_{m_s+\delta_s} = \overline{1,n+2}$, $\delta_s \in Z_0^+$, $s = \overline{1,\kappa}$, $l_{m_2} < l_{m_3} < ... < l_{m_\kappa}$, κ is the number of cycles.

We denote by υ the set of ordered pairs corresponding to decomposition (21), i.e.,

$$\{(n+2,1)(1,l_{m_1}),...,(l_{m_1+\delta_1}, n+2),(l_{m_2}, l_{m_2+1}),...,(l_{m_2+\delta_2}, l_{m_2}),...,(l_{m_\kappa}, l_{m_\kappa+1}),...,(l_{m_\kappa+\delta_\kappa}, l_{m_\kappa})\},$$

also, denote by υ' that the subset υ in the form

$$\{(1,l_{m_1}),...,(l_{m_1+\delta_1}, n+2),(l_{m_2}, l_{m_2+1}),...,(l_{m_2+\delta_2}, l_{m_2}),...,(l_{m_\kappa}, l_{m_\kappa+1}),...,(l_{m_\kappa+\delta_\kappa}, l_{m_\kappa})\}.$$

Further, we denote the bijective mapping from the set $\{1,2,...,n+1\}$ to σ' by ρ', and also by $A_{\rho'}$ the set of all mappings ρ' that define.

Lemma 4. *Let $\{\vec{x}_l\}_{l=1}^{n+2}$ be a sequence of vectors $\vec{x}_1,\vec{x}_2,...,\vec{x}_{n+1},\vec{x}_{n+2}$, the first n of which are strongly linearly independent. If the product $\omega_1\omega_2\cdot...\cdot\omega_{n+1}$ is equal to one of the values ± 1; $\pm i$; $\pm j$; $\pm k$, then the relation*

$$F(\vec{x}_1,...,\vec{x}_{n+1}, \vec{x}_{n+2}) = \sum_{\rho'_\tau \in A_{\rho'}} (-1)^{n+1-\kappa} c^{\omega}_{\rho'_\tau} F^{\omega_1,\omega_2...,\omega_{n+1}}_{\rho'_\tau} = 0 \quad (22)$$

is valid for the sequence $\{\vec{x}_l\}_{l=1}^{n+2}$, where

$$c^{\omega}_{\rho'} = (-1)^q sign\{\omega_1\cdot\omega_2\cdot...\cdot\omega_{n+1}\} \times \\ \times sign\left\{\Omega_{\omega_1}\left(\vec{x}_{l_{m_{s_1}}}, \vec{x}_{l'_{m_{s_1}}}\right)\Omega_{\omega_2}\left(\vec{x}_{l_{m_{s_2}}}, \vec{x}_{l'_{m_{s_2}}}\right)\cdot...\cdot\Omega_{\omega_{n+1}}\left(\vec{x}_{l_{m_{s_{n+1}}}}, \vec{x}_{l'_{m_{s_{n+1}}}}\right)\right\},$$

$q-$ is the number of imaginary units in the product $\omega_1 \cdot \omega_2 \cdot \ldots \cdot \omega_{n+1}$, $\omega_\ell \in \{1,\ i,\ j,\ k\}$, $\left\{l_{m_s}, l'_{m_s}\right\} = \left(\rho'_\tau\right)^{-1}(m_s)$, $l_{m_s} < l'_{m_s}$, $\tau = \overline{1, n+1!}$, $m_s = \overline{1, n+1}$.

Lemma 4 proves by Proposition 3, formula (14), and the definition of the bijective mapping $\rho'_\tau \in A_{\rho'}$. From in above theorem and lemmas, we obtain a corrollary the following:

Corollary 2 *System (10) is a complete table of typical basic invariants for the group $G = \mathfrak{Sp}(4n)$ and the statements of Lemmas 1-4 are represented the relations between of them.*

The differential field $\mathfrak{Sp}\,(4n)$ −invariant rational functions

In this section, we study the differential ring that the corresponding to he ring $R\left[\vec{x}_1, \vec{x}_2, ..., \vec{4n}\right]^{\mathfrak{Sp}(4n)}$. Also, we solve the problems describing the system of d-generators and finding the relations in between of them.

Let $\mathbb{K}$ be a commutative ring and d a derivation in $\mathbb{K}$, i.e.,

$$d(x+y) = d(x) + d(y),\ \ d(x \cdot y) = d(x) \cdot y + x \cdot d(y)$$

for any $x, y \in \mathbb{K}$.

It is known ([9]) that a derivation d in an integral domain $\mathbb{K}$ admits a unique extension to a derivation of the corresponding field of fractions.

A commutative ring $\mathbb{K}$ with unity (respectively, a field $\mathbb{P}$) in which a fixed derivation is specified is called *a differential ring* (d-ring), (respectively *a differential field*, d-field). A subfield $\mathbb{F}$ in a d-field $\mathbb{P}$ is called a *d-subfield* if $d(\mathbb{F}) \subset \mathbb{F}$.

We will use the following examples of d-rings and d-fields.

We fix a natural number $4n \in N$ and consider the ring of polynomials of countable number of variables

$$x_1^{(0)}, x_2^{(0)}, ..., x_{4n}^{(0)}, x_1^{(1)}, x_2^{(1)}, ..., x_{4n}^{(1)}, ..., x_1^{(r)}, x_2^{(r)}, ..., x_{4n}^{(r)}, ...,\ \ r = 0, 1, 2, ...$$

of the form

$$R\left[x_1^{(0)}, x_2^{(0)}, ..., x_{4n}^{(0)}, x_1^{(1)}, x_2^{(1)}, ..., x_{4n}^{(1)}, ..., x_1^{(r)}, x_2^{(r)}, ..., x_{4n}^{(r)}, ...\right]$$

with coefficients from the field of real numbers R; we denote this ring by $R\{\vec{x}\}$ (we assume that $x_l = x_l^{(0)}, l = \overline{1, 4n}$). We set $d\left(x_l^{(r)}\right) = x_l^{(r+1)},\ d(c) = 0, c \in R$, for all $l = \overline{1, 4n}, r \in Z_0^+$. The mapping d can be uniquely extended to a differentiation $\mathfrak{d}$ in the ring $R\{\vec{x}\}$. Then this ring becomes a differential ring, its elements are called *d-polynomials*; we denote them $f\{\vec{x}\}$, where $\vec{x} = \{x_l\}_{l=1}^{4n} \in V$.

We denote by $R\langle\vec{x}\rangle$ the field of fractions for the ring by $R\{\vec{x}\}$, i.e., $R\langle\vec{x}\rangle$ is the field of all rational functions of the same variables $x_l^{(r)},\ r \in Z_0^+, l = \overline{1, 4n}$. Also, the differentiation $\mathfrak{d}$ can be naturally extended from the ring $R\{\vec{x}\}$ to a differentiation on the field $R\langle\vec{x}\rangle$. Then this field becomes a differential field, and elements of the d-field $R\langle\vec{x}\rangle$ are called d-rational functions; we denote them $f\langle\vec{x}\rangle$.

Let G be a subgroup of the group $GL(4n, R)$.

Definition 6. A differential polynomial $f\{\vec{x}\}$ (respectively, a d-rational function $f\langle\vec{x}\rangle$) is said to be $G-$invariant if

$$f\{\vec{x}g\} = f\{\vec{x}\},\ \ (f\langle\vec{x}g\rangle = f\langle\vec{x}\rangle)$$

for all $g \in G$.

The set of all $G-$invariant d-polynomials (respectively, $G-$invariant d-rational functions) is denoted by $R\{\vec{x}\}^G$ (respectively, $R\langle\vec{x}\rangle^G$). It is known that $R\{\vec{x}\}^G \subset R\{\vec{x}\}$ (respectively, $R\langle\vec{x}\rangle^G \subset R\langle\vec{x}\rangle$).

Let the set $\Sigma' = \{\varepsilon'_l\}_{l \in L}$ is consisted from elements of $R\{\vec{x}\}^G$, where L is a set in finite number, the ordered of natural number.

A subset Σ' of $R\langle\vec{x}\rangle^G$ is called *a generating system* of the d-field $R\langle\vec{x}\rangle^G$ if an arbitrary element $f\langle\vec{x}\rangle$ of $R\langle\vec{x}\rangle^G$ can be generating by applying a finite number of operations of the d-field $R\langle\vec{x}\rangle^G$ to the elements of Σ', and the elements of Σ' are called *d-generators* of the d-field $R\langle\vec{x}\rangle^G$.

Elements $\{\varepsilon'_1, \varepsilon'_2, ..., \varepsilon'_s\}$ of Σ' are said to be *$d-$ algebraical dependent* over d-field $R\langle\vec{x}\rangle^G$ if there exists a non zero d-polynomial $P\{y_1, y_2, ..., y_s\} \in R\{\vec{x}\}$ such that $P\{\varepsilon'_1, \varepsilon'_2, ..., \varepsilon'_s\} = 0$. Otherwise, the system of elements $\varepsilon'_1, \varepsilon'_2, ..., \varepsilon'_s$ is said to be $d-$algebraically independent over $R\langle\vec{x}\rangle^G$. Also, a finite system of d-generators in $R\langle\vec{x}\rangle^G$ that is d-algebraically independent is called *d-rational basis* of the d-generators of the d-field $R\langle\vec{x}\rangle^G$.

We consider the following problem constructing a finite system of d-generators of the d-field $R\langle\vec{x}\rangle^G$ in case $G=\mathfrak{Sp}(4n)$.

Theorem 2. *A system of d-generators of the d-field* $R\langle\vec{x}\rangle^{\mathfrak{Sp}(4n)}$ *is formed be the polynomials*

$$\Omega_1\left(\vec{x}^{(r)},\vec{x}^{(r)}\right),\ \Omega_{\alpha_1}\left(\vec{x}^{(r)},\vec{x}^{(r+1)}\right),\left(r=\overline{0,n-1},\ \alpha_1\in\{i,j,k\}\right). \tag{23}$$

Proof. To prove Theorem 2, we use he following statements and propositions.

Statement 1. *Any* $\mathfrak{Sp}(4n)-$ *invariant d-rational function is the ratio of two* $\mathfrak{Sp}(4n)-$*invariant d-polynomials.* This statement follows from Proposition 1 in [7] (see, also the proof of Theorem 2.1.1 in [9]).

Statement 2. *Any* $\mathfrak{Sp}(4n)-$ *invariant d-polynomial is represented by the d-polynomials*

$$\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right),\ (\alpha\in\{1,i,j,k\}).$$

Statement 2 is differential analogy of Theorem 1.

Statement 3. *If* $\alpha\in\{1,i,j,k\}$ *and* $\vec{x}\in V$*, then the following affirmations are hold for the* $\mathfrak{Sp}(4n)-$*invariant d-polynomials* $\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$:

i_1) any d-polynomial $\Omega_1\left(\vec{x}^{(l)},\vec{x}^{(m)}\right),\ l,m\in Z_0^+$ is expressed in terms of the d-polynomials $\Omega_1\left(\vec{x}^{(r_1)},\vec{x}^{(r_1)}\right)$ with use of finite number of operations in the d-ring $R\{\vec{x}\}^{\mathfrak{Sp}(4n)}$, where $r_1\le\left[\frac{l+m}{2}\right]$;

i_2) any d-polynomial $\Omega_1\left(\vec{x}^{(r_1)},\vec{x}^{(r_1)}\right)$ is expressed d-rational by the d-polynomials $\Omega_1\left(\vec{x}^{\left(r_1'\right)},\vec{x}^{\left(r_1'\right)}\right)$, where $r_1\ge 4n,\ r_1'=\overline{0,4n-1}$;

i_3) any d-polynomial $\Omega_i\left(\vec{x}^{(l)},\vec{x}^{(m)}\right),\ l,m\in Z_0^+,\ l<m$ is expressed in terms of the d-polynomials $\Omega_i\left(\vec{x}^{(r_2)},\vec{x}^{(r_2+1)}\right)$ with the use of finite number of operations in the d-ring $R\{\vec{x}\}^{\mathfrak{Sp}(4n)}$, where $l+m\ge 2r_2+1$;

i_4) any d-polynomial $\Omega_i\left(\vec{x}^{(r_2)},\vec{x}^{(r_2+1)}\right),\ r_2\ge 4n$ is expressed d-rationally in term of the d-polynomials $\Omega_i\left(\vec{x}^{\left(r_2'\right)},\vec{x}^{\left(r_2'+1\right)}\right)$, where $r_2'=\overline{0,4n-1}$;

i_5) any d-polynomials $\Omega_j\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$ and $\Omega_k\left(\vec{x}^{(l)},\vec{x}^{(m)}\right),\ l,m\in Z_0^+,\ l<m$ are expressed in terms of the d-polynomials $\Omega_j\left(\vec{x}^{(r_3)},\vec{x}^{(r_3+1)}\right)$ and $\Omega_k\left(\vec{x}^{(r_3)},\vec{x}^{(r_2+1)}\right)$ with the use of finite number of operations in the d-ring $R\{\vec{x}\}^{\mathfrak{Sp}(4n)}$, where $l+m\ge 2r_3+1$;

i_6) any d-polynomial $\Omega_j\left(\vec{x}^{(r_3)},\vec{x}^{(r_3+1)}\right)$ and $\Omega_k\left(\vec{x}^{(r_3)},\vec{x}^{(r_3+1)}\right),\ r_3\ge 4n$ is expressed d-rationally in term of the d-polynomials $\Omega_j\left(\vec{x}^{\left(r_3'\right)},\vec{x}^{\left(r_3'+1\right)}\right)$ and $\Omega_k\left(\vec{x}^{\left(r_3'\right)},\vec{x}^{\left(r_3'+1\right)}\right)$ where $r_3'=\overline{0,4n-1}$.

Parts i_1) and i_2) of Statement 3 have been proved by Aripov R.G [10] and Xadjiyev Dj [7] in case generally; parts i_3) and i_4) was proved by Muminov K.K [8]; parts i_5) and i_6) are follow by applying the equalities

$$\Omega_j\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)=\Omega_i\left(\vec{x}^{(l)}A_1,\vec{x}^{(m)}A_1\right)\text{ and }\Omega_k\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)=\Omega_i\left(\vec{x}^{(l)}A_2,\vec{x}^{(m)}A_2\right)$$

to the proof of parts i_3) and i_4).

Note 2. Since the equality $\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)=-\Omega_\alpha\left(\vec{x}^{(m)},\vec{x}^{(l)}\right)$ is hold, the case $l>m$ of parts $i_3)-i_6$) follows from those case $l<m$.

From Statements 1-3, we get the following corollary for the $\mathfrak{Sp}(4n)-$invariant d-polynomial $\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$.

Corollary 3. *Any* $\mathfrak{Sp}(4n)-$*invariant d-rational function is expressed d-rationally with d-polynomials*

$$\Omega_1\left(\vec{x}^{(r')},\vec{x}^{(r')}\right),\ \ \Omega_{\alpha_1}\left(\vec{x}^{(r')},\vec{x}^{(r'+1)}\right),\ \ r'=\overline{0,4n-1},\ \ \alpha_1\in\{i,j,k\}. \tag{24}$$

It follows from Corollary 2 that to prove Theorem 2 it suffices to show that elements of system (24) will be d-rationally expressed by elements of system (23). In other words, we study the problem of minimizing the number of elements of the system (24). To do this, we widely use the following propositions:

Proposition 4. *For any non-zero* $\mathfrak{Sp}(4n)-$*invariant d-polynomial* $\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$, $(\alpha\in\{1,i,j,k\})$, *the following equality holds:*

$$d\left[\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)\right]=\Omega_\alpha\left(\vec{x}^{(l+1)},\vec{x}^{(m)}\right)+\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m+1)}\right),(\alpha\in\{1,i,j,k\})\,; \qquad (25)$$

Proposition 5. *For any sequence of the strong linear independent vectors* $\vec{x}$, $\vec{x}^{(1)}$, ..., $\vec{x}^{(n-1)}$ *in* V, *the following relation is true:*

$$F\left(\vec{x},\,\vec{x}^{(1)},\,...,\,\vec{x}^{(n-1)}\right)=\sum_{\rho_\tau\in A_\rho}(-1)^{n-\kappa}c^\alpha_{\rho_\tau}F^{\alpha_1,\alpha_2,...,\alpha_n}_{\rho_\tau}\neq 0; \qquad (26)$$

where $\alpha_1\cdot\alpha_2\cdot...\cdot\alpha_n=\pm1$, $\alpha_s\in\{1,\ i,\ j,\ k\}$,

$$c^\alpha_{\rho_\tau}=(-1)^q sign\left\{\alpha_1\cdot\alpha_2\cdot...\cdot\alpha_n\right\}\times sign\left\{\Omega_{\alpha_1}\cdot\Omega_{\alpha_2}\cdot...\cdot\Omega_{\alpha_n}\right\};$$

Proposition 6. *Let be a set* $\{\vec{x},\,\vec{x}^{(1)},\,...,\,\vec{x}^{(n-1)}\}$ *of strong linearly independent vectors in* V. *Then, the following relation*

$$F\left(\vec{x},\,\vec{x}^{(1)},\,...,\,\vec{x}^{(n)}\right)=\sum_{\rho_\tau\in A_\rho}(-1)^{n+1-\kappa}c^\alpha_{\rho_\tau}F^{\alpha_1,\alpha_2,...,\alpha_{n+1}}_{\rho_\tau}=0 \qquad (27)$$

is true for any sequence of vectors $\vec{x}$, $\vec{x}^{(1)}$, ..., $\vec{x}^{(n)}$ *in* V, *where* $\alpha_1\cdot\alpha_2\cdot...\cdot\alpha_{n+1}=\pm1$, $\alpha_s\in\{1,\ i,\ j,\ k\}$,

$$c^\alpha_{\rho_\tau}=(-1)^q sign\left\{\alpha_1\cdot\alpha_2\cdot...\cdot\alpha_n\right\}\times sign\left\{\Omega_{\alpha_1}\cdot\Omega_{\alpha_2}\cdot...\cdot\Omega_{\alpha_n}\right\};$$

Proposition 7. *Let be a set* $\{\vec{x},\,\vec{x}^{(1)},\,...,\,\vec{x}^{(n-1)}\}$ *of strong linearly independent vectors in* V. *Then, the following relation*

$$F\left(\vec{x},\,\vec{x}^{(1)},\,\vec{x}^{(2)},\,...\,,\,\vec{x}^{(n+1)}\right)=\sum_{\rho'_\tau\in A_{\rho'}}(-1)^{n+1-\kappa}c^\omega_{\rho'_\tau}F^{\omega_1,\,\omega_2,...,\omega_{n+1}}_{\rho'_\tau}=0 \qquad (28)$$

is true for the $\vec{x}$, $\vec{x}^{(1)}$, ..., $\vec{x}^{(n)}$, $\vec{x}^{(n+1)}$ *in* V, *where* $w_1\cdot w_2\cdot...\cdot w_n=\beta$, $\beta\in\{\pm1,\ \pm i,\ \pm j,\ \pm k\}$,

$$c^w_{\rho_\tau}=(-1)^q sign\left\{w_1\cdot w_2\cdot...\cdot w_{n+1}\right\}\times sign\left\{\Omega_{w_1}\cdot\Omega_{w_2}\cdot...\cdot\Omega_{w_{n+1}}\right\}.$$

Proposition 4 follows from definition of the operation differential; Propositions 5-7 represents the differential analogy of Lemmas 2-4.

We first minimize the number of d-polynomials $\Omega_1\left(\vec{x}^{(r')},\vec{x}^{(r')}\right)$, $(r'=\overline{0,4n-1})$ using the above propositions and statements. To do this, we use the method of mathematical induction:

Step 1. Let be $r'=\overline{0,n-1}$. Then the $d-$polynomials $\Omega_1\left(\vec{x}^{(r')},\vec{x}^{(r')}\right)$ is elements of system (23);

Step 2. Let be $r'=n$. There expressing equality (27) of Proposition 6 in the form

$$F\left(\vec{x},...,\vec{x}^{(n)}\right)=F_1\left(\vec{x},...,\vec{x}^{(n)}\right)+\Omega_1\left(\vec{x}^{(n)},\vec{x}^{(n)}\right)F\left(\vec{x},...,\vec{x}^{(n-1)}\right)=0,$$

we obtain the following

$$\Omega_1\left(\vec{x}^{(n)},\vec{x}^{(n)}\right)=-\frac{F_1\left(\vec{x},...,\vec{x}^{(n)}\right)}{F\left(\vec{x},...,\vec{x}^{(n-1)}\right)}, \qquad (29)$$

where the expression $F_1\left(\vec{x},...,\vec{x}^{(n)}\right)$ will be the sum of such terms of the expression $F\left(\vec{x},...,\vec{x}^{(n)}\right)$ that it doesn't contain d-polynomial $\Omega_1\left(\vec{x}^{(n)},\vec{x}^{(n)}\right)$; it is known from Proposition 5 that $F\left(\vec{x},...,\vec{x}^{(n-1)}\right)$ isn't equal to zero; therefore the fraction is well defined, which on the right side of formula (29); also the expression$F_1\left(\vec{x},...,\vec{x}^{(n)}\right)$ and $F\left(\vec{x},...,\vec{x}^{(n-1)}\right)$ consist of d-polynomials $\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$, $l,m=\overline{0,n}$,
$\alpha\in\{1,\,i,\,j,\,k\}$; In this case, under the conditions $i_1)$, $i_2)$, $i_3)$ of Statements 3, we have that the $\Omega_1\left(\vec{x}^{(r_1)},\vec{x}^{(r_1)}\right)$, $\Omega_{\alpha_1}\left(\vec{x}^{(r_2)},\vec{x}^{(r_2+1)}\right)$, $(\alpha_1\in\{i,j,k\})$ are expressed d-rationally in terms of elements of System (23), since
$l+m\le\max\{l+m\}=2n-1$,

$$r_1\le\left[\frac{2n-1}{2}\right]=\left[n-\frac{1}{2}\right]=n-1\Rightarrow r_1=\overline{0,n-1};\ \ 2r_2+1\le 2n-1\Rightarrow r_2\le n-1\Rightarrow r_2=\overline{0,n-1}.$$

Hence, the expressions $F_1\left(\vec{x},...,\vec{x}^{(n)}\right)$ and $F\left(\vec{x},...,\vec{x}^{(n-1)}\right)$ are also expressed d-rationally in terms of elements of System (23). From this and the equality (29) follows that d-polynomial $\Omega_1\left(\vec{x}^{(n)},\vec{x}^{(n)}\right)$ is also expressed d-rationally by the elements of System (23);

Now, we shall show that the $\mathfrak{Sp}(4n)-$invariant d-polynomials $\Omega_{\alpha_1}\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right)$, $(\alpha_1=\{i,\,j,\,k\})$ are expressed d-rationally in terms of elements of the system (23). To do this we use the equality (26), as a result we have the system,

$$\begin{cases} \sum\limits_{\rho'_\tau\in A_{\rho'}} (-1)^{n+1-\kappa} c^{\alpha}_{\rho'_\tau} F^{\alpha_1\cdot\ldots\cdot\alpha_{n+1}}_{\rho'_\tau}=0, & \alpha_1\cdot\ldots\cdot\alpha_{n+1}=\pm 1;\\ \sum\limits_{\rho'_\tau\in A_{\rho'}} (-1)^{n+1-\kappa} c^{\beta}_{\rho'_\tau} F^{\beta_1\cdot\ldots\cdot\beta_{n+1}}_{\rho'_\tau}=0, & \beta_1\cdot\ldots\cdot\beta_{n+1}=\pm i;\\ \sum\limits_{\rho'_\tau\in A_{\rho'}} (-1)^{n+1-\kappa} c^{\gamma}_{\rho'_\tau} F^{\gamma_1\cdot\ldots\cdot\gamma_{n+1}}_{\rho'_\tau}=0, & \gamma_1\cdot\ldots\cdot\gamma_{n+1}=\pm j;\\ \sum\limits_{\rho'_\tau\in A_{\rho'}} (-1)^{n+1-\kappa} c^{\nu}_{\rho'_\tau} F^{\nu_1\cdot\ldots\cdot\nu_{n+1}}_{\rho'_\tau}=0, & \nu_1\cdot\ldots\cdot\nu_{n+1}=\pm k \end{cases} \tag{30}$$

where $\alpha_\ell,\beta_\ell,\,\gamma_\ell,\nu_\ell\in\{1,i,j,k\}$, $l=\overline{1,\,n+1}$.

Let us system (30) with respect to the polynomials $\Omega_\alpha\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right)$, $(\alpha\in\{1,i,\,j,\,k\})$. To do this, denote by ρ''_τ a bijective mapping that satisfies the conditions $\left(\rho'_\tau\right)^{(-1)}(n+1)=\{n,n+1\}$, and we obtain the following

$$F^{\omega_1,\ldots,\omega_{n+1}}_{\rho''_\tau}=F^{\omega_1,\ldots,\omega_n}_{\rho''_\tau}\Omega_{\omega_{n+1}}\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right)$$

where $\omega_\ell\in\{\alpha_\ell,\beta_\ell,\,\gamma_\ell,\nu_\ell\}$;
Also, we introduce notations in the following:

$$z_1=\Omega_1\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right),\quad z_2=\Omega_i\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right),\quad z_3=\Omega_j\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right),\quad z_4=\Omega_k\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right)$$

$$\begin{aligned} D_1&=\sum_{\rho''_\tau\in A_{\rho'}}(-1)^{n+1-\kappa}c^{\alpha}_{\rho''_\tau}F^{\alpha_1,\ldots,\alpha_n}_{\rho''_\tau},(\alpha_1\cdot\ldots\cdot\alpha_n=\pm 1);\\ D_2&=\sum_{\rho''_\tau\in A_{\rho'}}(-1)^{n+1-\kappa}c^{\beta}_{\rho''_\tau}F^{\beta_1,\ldots,\beta_n}_{\rho''_\tau},(\beta_1\cdot\ldots\cdot\beta_n=\pm i);\\ D_3&=\sum_{\rho''_\tau\in A_{\rho'}}(-1)^{n+1-\kappa}c^{\gamma}_{\rho''_\tau}F^{\gamma_1,\ldots,\gamma_n}_{\rho''_\tau},(\gamma_1\cdot\ldots\cdot\gamma_n=\pm j);\\ D_4&=\sum_{\rho''_\tau\in A_{\rho'}}(-1)^{n+1-\kappa}c^{\nu}_{\rho''_\tau}F^{\nu_1,\ldots,\nu_n}_{\rho''_\tau},(\nu_1\cdot\ldots\cdot\nu_n=\pm k); \end{aligned}$$

Moreover, we denote by $N_1,\,N_2,\,N_3,\,N_4$ the sum of the products $F^{\omega_1,\ldots,\omega_{n+1}}_{\rho'_\tau}$ corresponding to the mapping $\rho'_\tau\in A_{\rho'}$ that satisfies the conditions $\left(\rho'_\tau\right)^{(-1)}(n+1)\neq\{n,n+1\}$. From these notations and from system (30), we have a system of the following form:

$$\begin{cases} -D_1z_1+D_2z_2+D_3z_3+D_4z_4=N_1\\ D_2z_1+D_1z_2+D_4z_3-D_3z_4=N_2\\ D_3z_1-D_4z_2+D_1z_3+D_2z_4=N_3\\ D_4z_1+D_3z_2-D_2z_3+D_1z_4=N_4. \end{cases} \tag{31}$$

It is easily shown that a determinant of coefficient matrix of system (31) is defined by formula

$$\Delta'=-\left(D_1^2+D_2^2+D_3^2+D_4^2\right)^2,$$

where Δ' is determinant of coefficient matrix.

Hence, the equal $\Delta'=0$ is true if and only if $D_1=D_2=D_3=D_4=0$. According to Lemme 1, at least one of the coefficients $D_1,\,D_2,\,D_3,\,D_4$ is nonzero. Hence, $\Delta'\neq 0$, and system (31) has a unique solution. In this case, we can be represented by d-polynomials $\Omega_\omega\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right)$, $(\omega\in\{1,i,j,k\})$ in terms of $D_1,\,D_2$, $D_3,\,D_4$ and N_t $\left(t=\overline{1,4}\right)$ in a unique form. It is easily shown that the conditions $l+m\leq 2n$ is hold for the d-polynomials in these expressions. From this and from i_3) it is follows that d-polynomials $\Omega_\omega\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$, $(\omega\in\{1,\ i,\ j,\ k\})$ are expressed d-rationally in terms of the elements of system (23). This shown that the d-polynomials $\Omega_\omega\left(\vec{x}^{(n)},\vec{x}^{(n+1)}\right)$, $(\omega\in\{1,i,j,k\})$ are also expressed d-rationally by the elements of system (23).

Step 3. Let be $r'=n+s,(s=\overline{2,3n-2})$. In this case, we assume that the statement of Theorem 2 is true;

Now, we prove that the statement of Theorem 2 is true by using the above assumption, for $r'=n+s+1=4n-1$. To do this, we write of equality (27) in Proposition 6 for a set of vectors $\vec{x}^{(3n-1)},...,\vec{x}^{(4n-1)}$, and have to the following

$$F\left(\vec{x}^{(3n-1)},...,\vec{x}^{(4n-1)}\right)=F_1\left(\vec{x}^{(3n-1)},...,\vec{x}^{(4n-1)}\right)+\Omega_1\left(\vec{x}^{(4n-1)},\,\vec{x}^{(4n-1)}\right)F\left(\vec{x}^{(3n-1)},...,\vec{x}^{(4n-2)}\right)=0$$

From the above equality, it follows

$$\Omega_1\left(\vec{x}^{(4n-1)},\,\vec{x}^{(4n-1)}\right)=-\frac{F_1\left(\vec{x}^{(3n-1)},...,\vec{x}^{(4n-1)}\right)}{F\left(\vec{x}^{(3n-1)},...,\vec{x}^{(4n-2)}\right)},\quad\left(F\left(\vec{x}^{(3n-1)},...,\vec{x}^{(4n-2)}\right)\neq 0.\right)$$

The d-polynomials $\Omega_\alpha\left(\vec{x}^{(l)},\vec{x}^{(m)}\right)$ in the expressions $F_1\left(\vec{x}^{(3n-1)},...,\vec{x}^{(4n-1)}\right)$ and $F\left(\vec{x}^{(3n-1)},...,\vec{x}^{(4n-2)}\right)$, are expressed d-rationally in terms of the elements of system (23) under Statement 3 and the assumption, because $l,m=\overline{3n-1,4n-1}$. This implies that the d-polynomials $\Omega_1\left(\vec{x}^{(4n-1)},\,\vec{x}^{(4n-1)}\right)$ is also expressed d-rationally in terms of the elements of system (23);

As above, it can be shown that the d-polynomials $\Omega_{\alpha_1}\left(\vec{x}^{(4n-1)},\,\vec{x}^{(4n)}\right)$ are also expressed d-rationally of the elements of system (23) in accordance with Statement 3 and assumption. To do this, it suffices to repeat the calculation for d-polynomial $\Omega_{\alpha_1}\left(\vec{x}^{(n)},\,\vec{x}^{(n+1)}\right)$ by a set of vectors $\vec{x}^{(3n-1)},...,\vec{x}^{(4n)}$. Hence, from the principle of Mathematical Induction, it follows that the d-polynomials $\Omega_1\left(\vec{x}^{(r')},\,\vec{x}^{(r')}\right)$, $\Omega_{\alpha_1}\left(\vec{x}^{(r')},\,\vec{x}^{(r'+1)}\right)$ are expressed d-rationally of the elements of the system (23) for all values of $r'=\overline{0,4n-1}$. That is exactly what we wanted to show. Theorem 2 is proved. □

In conclusion, we can state the following corollary from Theorem 2.

Corollary 4. *The d-field of* $\mathfrak{Sp}(4n)-$*invariant d-rational functions over R has a finite number of d-generators, and their number is equal to* $4n$.

References

1. Vinberg E. B., Popov V. L. Theory of invariants. *Sovremn. Probl. Matem. Fundam. Napravl* 55, 1998, pp.137-309. [in Russian]

2. Diendone J. A., Carrell J. B. Invariant theory. Academik Press. 1971.

3. Nagata M. On the fourteen problem of Hilbert. Lectures in Tata institute of Fundamental Research. Bombey. 1963.

4. Mamford D., Forgaty J. Geometric invariant theory. Berlin: springer-Verlag.1982.

5. Gilbert D. Selected works. Volume 1 Theory of invariants. Theory of numbers. Algebra. Geometry. Fundamentals of mathematics. M. Faktorial. 1998. [in Russian]

6. Weyl H. The classical groups. Their invariants and representation. Princeton. Univ. Press. 1997.

7. Xadjiyev Dj. Application of the theory of invariants to the differential geometry of curves. Tashkent: Fan, 1988. [in Russian]

8. Muminov K. K. Equivalence of paths with respect to the action of a symplectic group.*Izv. Vyssh. Uchebn. Zaved. Mat* 7, 2002. pp 27-38. [in Russian]

9. Muminov K. K., Chilin. V. .I. Equivalence of curves in finite-dimensional spaces. Lambert Academic Publishing. Deutchland 2015. [in Russian]

10. Aripov R. G. Generators of the differential field of invariant differential rational path functions for the action of real orthogonal groups in C^n.*Uzbek mathematical journal*, 1, 1992, pp.3-7.[in Russian]

11. Khadjiyev Dj., Oren I., Peksen O. Generating systems of differential invariants and the theorem on extense for curves in the pseudo-Euclidean geometry.*Turkish Journal of Mathematics*, 37, 2013, pp.80-94.

12. Khadjiyev Dj. Complete systems of differential invariants of vector fields in a euclidean space. *Turkish Journal of Mathematics*, 34, 2010, pp.543-559.

13. Chilin. V. I., Muminov K. K. Equivalence of paths in Galilean geometry. *Journal of Mathematical Sciences* 245(3), 2020, pp. 297-309.

14. Muminov K. K., Gafforov R. A. Generators of the differential field of invariant rational functions of a finite number of paths under the action of a special pseudo-orthogonal group. *Reports of the Academy of Sciences of the Republic of Uzbekistan.* 11, 2011, b.12-13. [in Russian]

15. Muminov K. K., Chilin. V. .I. A transcendence basis in the differential field of invariants of pseudo-Galilean group. *Russian Mathematics.(Iz. Vuz)*, 63(3), 2019, pp 15-24.

16. Iwahori N Some remarks on tensor invariants of $O(n)$, $U(n)$, $Sp(n)$. *Journal of the Mathematical Society of Japan*, 10(2), 1958. pp. 145-160

17. Bernig A. Invariant valuations on quaternionic vector spaces. *arxiv:1005.3654v1.[math.DC.]*, 20 may, 2010 Mat, 2, 2002, pp. 27-38.

18. Chevalley.C Iheory of Lie groups. i. Princeton., Princeton University Press. 1946.

19. Kostrikin A. I., Manin Yu. I. Linear algebra and geometry. Moscow: Nauka, 1986. [in Russian]

20. Naymark M. A. Group representation theory. Moscow: Nauka, 1976. [in Russian]

21. Kirchey I. I. Cramer's rule for quaternionic systems of linear equations. *Fundamentalnaya i prikladnaya matematika*, 13(4), 2007, pp.67-94. [in Russian]

22. Muminov K. K., Juraboyev S. S. Equivalence of paths with respect to the action of the group of real representations of quaternion numbers. *Scientific bulletin of Namangan state University*, 9, 2021, pp. 14-21. [in Uzbek]

23. Juraboyev S. S. Differential invariants with respect to the action of a group of real representation of the symplectic group. *Sarymsakov Readings: Abstracts of reports of the republican scientific conference with the participation of foreign scientists.* Tashkent. "Universitet". 2021. pp. 297-299. [in Uzbek]

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